Signals and Systems

16/04/2009, Thursday, 9:00-12:00

1 Fourier series/transform

(a) Consider the signal

$$f(t) = \frac{\sin^2(4t) - \sin^2(2t) - \sin^2(t)}{t^2}.$$

(i) Find and plot its Fourier transform.

(ii) Compute the integral

$$\int_{-\infty}^{\infty} f(t) \sin(wt)\, dt.$$

(b) Let $x(t)$ have Fourier transform $X(\omega)$ and let $y(t)$ be piecewise smooth periodic with fundamental frequency ω_0 and absolutely convergent Fourier series representation

$$y(t) = \sum_{k=-\infty}^{\infty} y_k e^{ik\omega_0 t}.$$

(i) What is the Fourier transform of $x(t)y(t)$?

(ii) Suppose that $X(\omega) = \text{trian}_1(\omega)$ and $y(t) = \cos(2t) - \cos(t)$. Sketch the spectrum of $x(t)y(t)$.

2 Generalized Fourier transform

Let $F(\omega)$ be the generalized Fourier transform of the signal $f(t)$. Verify that the differentiation rule

$$\mathcal{F}\{f'(t)\} = i\omega F(\omega)$$

holds for the signals

(a) $\mathbb{1}(t-2)$

(b) $\text{sgn}(t)$

(c) $e^{i\omega_0 t}$.

3 Laplace transform

(a) Let $f(t)$ be a causal signal with

$$f(t+T) = f(t)$$

for all $t \geq 0$ and for some $T > 0$. Show that its Laplace transform is given by

$$F(s) = \frac{\int_0^T f(t) e^{-st} dt}{1 - e^{-sT}} \quad \text{Re}(s) > 0.$$

(b) Determine the inverse Laplace transform of

$$F(s) = \frac{1 + e^{-s\pi}}{s^2 + 1} \quad \text{Re}(s) > 0$$

$$G(s) = \frac{1 - e^{-as}}{s(1 - e^{-bs})} \quad 0 < a < b \text{ and } \text{Re}(s) > 0.$$

4 Linear differential systems

(a) Consider the system

$$\begin{aligned} \dot{x}_1 &= ax_1 + bx_2 + u \\ \dot{x}_2 &= -bx_1 + ax_2 \\ \dot{x}_3 &= cx_3 + u \\ y &= x_1 + x_3. \end{aligned}$$

(i) Find the transfer function of the system.

(ii) For which values of (a, b, c) is this a BIBO-stable system?

(b) Let

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Find e^{At}.

5 Butterworth filters

(a) Show that the transfer function of the 3rd order Butterworth filter is given by

$$H_3(s) = \frac{1}{(s+1)(s^2+s+1)}.$$

(b) Find the corresponding impulse response for $H_3(s)$.

Hints

- The triangular pulse is defined by

$$\text{trian}_a(t) = \begin{cases} 1 - \frac{|t|}{a} & \text{if } |t| \leq a \\ 0 & \text{if } |t| > a \end{cases}$$

where a is a positive real number. Its Fourier transform is

$$F(\omega) = 4\frac{\sin^2(\frac{a\omega}{2})}{a\omega^2}.$$

- The Laplace transform of the function

$$f(t) = \sin(\omega t)$$

is given by

$$F(s) = \frac{\omega}{s^2 + \omega^2}$$

for $\text{Re}(s) > 0$.